

Energy extraction from Kerr black holes by rigidly rotating strings

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In this paper, we show that a rigidly rotating string can extract the rotational energy from a rotating black hole. We consider Nambu-Goto strings stationary with respect to a co-rotating Killing vector with an uniform angular velocity ω in the Kerr spacetime. We show that a necessary condition of the energy-extraction process is that an effective horizon on the string worldsheet, which corresponds to the inner light surface, is inside the ergosphere of the Kerr black hole and the angular velocity ω is less than that of the black hole Ω_h . Furthermore, we discuss global configurations of such strings in both of a slow-rotation limit and the extremal Kerr case.

I. INTRODUCTION

It is well known that the rotational energy of black holes can be extracted, and various mechanisms of the energy extraction have been proposed: the Penrose process caused by a particle fragmentation [1, 2], superradiance with amplifying incident waves for various fields [3–7], the Blandford-Znajek process in electromagnetic fields around a rotating black hole [8], and so on.¹ Extracting energy plays an important role in astrophysics as well as in general relativity.

In this paper, we show that the rotational energy can be extracted from a rotating black hole by a rigidly rotating string twining around it. Strings are interesting objects in several contexts; for example, in cosmology cosmic strings will be important probes of the early Universe, and in string theory, of course, strings themselves are so fundamental to construct the theories. The current system may be fascinating as a simple example of interacting systems of a string and a black hole and furthermore various extended objects and black objects [11–13]. In addition, this is expected to be a toy model of magnetic flux around a rotating black hole. As is clear from the stress-energy tensor of the Maxwell field, magnetic flux has magnetic tension. To study some phenomena in which magnetic tension is essential but magnetic pressure is not so significant, one may substitute magnetic flux with strings. Energy-extraction mechanism by strings from this perspective were discussed in Refs. [14, 15].

We consider stationary Nambu-Goto strings with respect to a co-rotating Killing vector characterizing a frame rotating with a constant angular velocity ω , which is a linear combination of stationary and axisymmetric Killing vectors, in the Kerr spacetime. Such rigidly rotating strings in stationary axisymmetric spacetimes were studied in Ref. [16, 17], where for $\omega = 0$, particularly, explicit analytic solutions were obtained. Here, we will focus on the induced geometry on the string worldsheet in order to reveal the nature of the energy extraction in detail.

In general, if a system is assumed to be rigidly rotating, the locus where its rotational velocity will exceed the speed of light emerges. The surface on which the velocity coincides with the speed of light is called as “light cylinder” or “light surface” more generally. In stationary cases, it becomes the stationary limit surface at which the norm of the co-rotating Killing vector is zero and beyond which the Killing vector becomes spacelike.

Similarly, for a rigidly rotating string such surface will be reflected in an effective Killing horizon on the induced geometry of the string worldsheet. Because dynamics of the string is governed by its induced metric, the effective horizon determined by the induced metric acts as a causal boundary for the stationary region on the worldsheet such that the stationary Killing vector is timelike with respect to the induced metric. However, whether an effective horizon exists or not depends on the actual configuration of the string determined by the equations of motion. If the string configuration does not reach the light surface, for instance, no effective horizon emerges on the string worldsheet. The aim of this paper is to explore a rigidly rotating string on which the effective horizon exists with an energy flow. This implies that the string should be regularly passing through the light surface and extending far beyond. It turns out that such strings are characterized by two parameters: the angular velocity and the angular momentum flux (i.e., torque).

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¹ For example, the magnetic Penrose process [9]. See also Ref. [10] and references therein.

The reason why the effective horizon is significant is as follows. Even if the string is extending far beyond, a segment of the string beyond the effective horizon can never affect causally the other segment remaining in the stationary region. This means that we should be unconcerned about the segment of the string beyond the effective horizon to solve string dynamics in the stationary region. The string is regarded as an “open” string, which has different end-points, in its causal patch where dynamics can be determined by initial conditions prepared in the stationary region. If there is energy flux on this “open” string, the energy can be transferred from the inner end-point near the rotating black hole to the outer end-point (or infinity) far from the black hole. Thus, we accomplish net energy transfer across the stationary region causally connected on the string worldsheet.

The paper is organized as follows. In Sec. II we examine the induced geometry of rigidly rotating strings and the conditions that there exists an effective horizon. As a result, we obtain a parameter space of the angular velocity and the angular momentum flux in which physically reasonable string configurations can exist and show that energy-extraction process occurs. We discuss global structures of such strings in Sec. III. In a slow-rotation approximation, where both of a black hole and a string are slowly rotating, we analytically study global solutions, and then in the extremal Kerr case we show various global configurations obtained by numerically solving the equations of motion. Unless otherwise specified, Newton’s constant G_N and the speed of light c are set to unity in this paper.

II. RIGIDLY ROTATING STRING IN THE KERR SPACETIME

A. Effective horizon and regularity condition

The metric of the Kerr spacetime in the Boyer-Lindquist coordinates is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{2Mr}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2, \quad (1)$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr. \quad (2)$$

The larger root of $\Delta(r) = 0$ yields the radius of the event horizon $r_h \equiv M + \sqrt{M^2 - a^2}$. The angular velocity of the event horizon is $\Omega_h \equiv a/(a^2 + r_h^2)$. The radius of the ergosphere, characterized by $g_{tt} = 0$, is $r_{\text{ergo}}(\theta) \equiv M + \sqrt{M^2 - a^2 \cos^2\theta}$. Note that the Greek indices, μ and ν , are used for spacetime components.

In this background, we consider the stationary rotating string of which the worldsheet is tangent to the co-rotating Killing vector

$$\xi^\mu = (\partial_t)^\mu + \omega(\partial_\phi)^\mu, \quad (3)$$

where a constant ω is the angular velocity of the rotating frame. Thus, the string is rigidly rotating with the angular velocity ω . This string is embedded as

$$\phi = \omega t + \varphi(\sigma), \quad r = r(\sigma), \quad \theta = \theta(\sigma), \quad (4)$$

where we have synchronized the time coordinate of the string worldsheet with t in the Boyer-Lindquist coordinates. The induced metric on the string worldsheet is given by

$$\begin{aligned} h_{ab}d\sigma^a d\sigma^b = & - \left[1 - \frac{2Mr}{\Sigma}(1 - \omega a \sin^2\theta)^2 - \omega^2(r^2 + a^2)\sin^2\theta \right] dt^2 \\ & + 2\varphi' \sin^2\theta \left[\omega(r^2 + a^2) - \frac{2Mr}{\Sigma}a(1 - \omega a \sin^2\theta) \right] dt d\sigma \\ & + \left[\Sigma \left(\frac{r'^2}{\Delta} + \theta'^2 \right) + \frac{(r^2 + a^2)^2 - a^2\Delta \sin^2\theta}{\Sigma} \sin^2\theta \varphi'^2 \right] d\sigma^2, \end{aligned} \quad (5)$$

where the prime denotes derivative with respect to σ . Note that the Latin indices, a and b , are used for the worldsheet components. This metric admits the stationary Killing vector $\xi^a = (\partial_t)^a$ that is induced from ξ^μ . Indeed, the Killing vectors ξ^μ and ξ^a with respect to the spacetime metric and the induced metric are directly related to each other as $\xi^a \partial_a X^\mu = (\partial_t)^\mu + \omega(\partial_\phi)^\mu = \xi^\mu$, where X^μ denote the embedding functions of the string. It turns out that the norm of ξ^a would vanish. A locus of such effective (Killing) horizons on the string worldsheet is determined by

$$F(r, \theta) \equiv -h_{tt} = 1 - \frac{2Mr}{\Sigma}(1 - \omega a \sin^2\theta)^2 - \omega^2(r^2 + a^2)\sin^2\theta = 0. \quad (6)$$

Its condition can be rewritten as

$$\Delta(r) = \frac{[a - (a^2 + r^2)\omega]^2}{(1 - a\omega \sin^2\theta)^2} \sin^2\theta. \quad (7)$$

This condition, indeed, implies that the norm of the Killing vector with respect to the spacetime metric vanish, namely $g_{\mu\nu}\xi^\mu\xi^\nu = 0$, as well as the induced metric. Therefore, the surfaces that $F(r, \theta) = 0$ represents are nothing but stationary limit surfaces in terms of the rigid rotation with the angular velocity ω . In general, $F(r, \theta) = 0$ has two positive roots in terms of r like a metric of a black hole with cosmological constant. The inner surface and the outer one correspond to so-called inner light sphere and outer light cylinder, respectively. Note that, if ω is sufficiently large, the two light surfaces will merge into one connected light surface. In this case the number of positive roots of $F(r, \theta) = 0$ can be less than two for a given θ .²

² We thank Chulmoon Yoo for pointing out it.

We consider the rigidly rotating Nambu-Goto string. The Nambu-Goto action for the string is

$$S = - \int dt d\sigma \sqrt{-h} = \int dt d\sigma \mathcal{L}, \quad (8)$$

where h is the determinant of h_{ab} , and we have defined

$$\mathcal{L} \equiv - \sqrt{F \Sigma \left(\frac{r'^2}{\Delta} + \theta'^2 \right) + \Delta \sin^2 \theta \varphi'^2}. \quad (9)$$

For simplicity, we have omitted an overall factor in the above Lagrangian density, because it is irrelevant to dynamics of the strings. It means that the tension of the strings has been set to unity, or dynamics of the strings with a unit tension has been described. Since this action does not depend on $\varphi(\sigma)$ explicitly, we have a conserved quantity given by

$$q \equiv \frac{\partial \mathcal{L}}{\partial \varphi'} = \frac{\Delta \sin^2 \theta}{\mathcal{L}} \varphi'. \quad (10)$$

The sign of q is directly connected with the sign of φ' because $\Delta \geq 0$ outside the event horizon: $q > 0$ for $\varphi' < 0$ and $q < 0$ for $\varphi' > 0$. This quantity is interpreted as the angular momentum flux flowing on the string worldsheet outwardly,³ and it is written as

$$\begin{aligned} q &= \sqrt{-h} T^{\sigma a} \partial_a X^\mu (\partial_\phi)^\nu g_{\mu\nu} \\ &= -\sqrt{-h} [h^{\sigma t} g_{t\phi} + (h^{\sigma t} \omega + h^{\sigma\sigma} \varphi') g_{\phi\phi}], \end{aligned} \quad (11)$$

where $T^{ab} = -h^{ab}$ is the energy-momentum tensor on the worldsheet for the Nambu-Goto string. Similarly, the outward energy flux is written as

$$\begin{aligned} \omega q &= -\sqrt{-h} T^{\sigma a} \partial_a X^\mu (\partial_t)^\nu g_{\mu\nu} \\ &= \sqrt{-h} [h^{\sigma t} g_{tt} + (h^{\sigma t} \omega + h^{\sigma\sigma} \varphi') g_{\phi t}]. \end{aligned} \quad (12)$$

Obviously, these angular momentum flux and energy flux are respectively associated with the Killing vectors in the Kerr spacetime, namely $(\partial_\phi)^\mu$ and $(\partial_t)^\mu$, so that they are conserved in terms of σ . Note that, if one wants to consider a general value μ of the tension rather than unity, one should replace q with μq .

Although the locus of the effective horizon is determined by $F(r, \theta) = 0$, whether the effective horizon does actually exist or not depends on the configuration of the string given by the equations of motion. Let us see a condition for the effective horizon to exist regularly on the string worldsheet. We can rewrite Eq. (10) as

$$\frac{r'^2}{\Delta} + \theta'^2 = \frac{\Delta \sin^2 \theta (\Delta \sin^2 \theta - q^2)}{q^2 \Sigma F} \varphi'^2. \quad (13)$$

The left-hand side of the above equation cannot be negative outside the event horizon, i.e., $\Delta > 0$. On the other hand, the denominator in the right-hand side can change its sign beyond the effective horizon $F(r, \theta) = 0$. Hence, the numerator must be zero at the effective horizon and must change its sign for the configuration of the string to extend regularly beyond the effective horizon. At the effective horizons $F(r, \theta) = 0$, we have the regularity condition

$$q^2 = \Delta \sin^2 \theta. \quad (14)$$

If the above condition is not satisfied at $F(r, \theta) = 0$, then the induced geometry becomes singular there and the locus determined by $F(r, \theta) = 0$ is a singularity rather than a horizon. Note that, if we regard φ as a “time” coordinate and $(dr/d\varphi, d\theta/d\varphi) = (r'/\varphi', \theta'/\varphi')$ as a “velocity,” we can interpret Eq. (13) as an equation of mechanical energy for a particle in an effective potential on two-dimensional curved space. In that sense the condition of Eq. (14) implies “turning-point” of the effective potential.

Now, we are interested in regular solutions of the string passing through the effective horizon. Therefore, we will require the string to satisfy the both conditions of Eqs. (6) and (14). Solving the two conditions outside the event

³ Here, “outward” means $r' > 0$.

horizon ($r > r_h$), we have two branches of solutions in terms of r (see Appendix A 1). The locus of the effective horizon is

$$r_{\text{eff}}^{\pm}(\omega, q) = \frac{M}{1 \pm q\omega} + \sqrt{\left(\frac{M}{1 \pm q\omega}\right)^2 - a(a \mp q)}, \quad (15)$$

and

$$\sin \theta_{\text{eff}}^{\pm} = |q|/[\Delta(r_{\text{eff}}^{\pm})]^{1/2}, \quad (16)$$

where $q[a - (a^2 + r^2)\omega] > 0$ is satisfied if the upper sign is chosen and $q[a - (a^2 + r^2)\omega] < 0$ is satisfied if the lower sign is chosen. Note that, for simplicity, we have restricted the domain to the interval $0 \leq \theta \leq \pi/2$ because the current system is symmetric under $\theta \rightarrow \pi - \theta$. Since the branches of r_{eff}^{\pm} are related to each other in $q \rightarrow -q$ such as $r_{\text{eff}}^{-}(\omega, q) = r_{\text{eff}}^{+}(\omega, -q)$, we shall focus on $r_{\text{eff}}^{+}(\omega, q)$ in what follows.

In order to satisfy $0 \leq \sin^2 \theta_{\text{eff}}^{+} \leq 1$, it turns out that ω lies in the intervals

$$\begin{aligned} \omega_0(q) < \omega \leq \omega_{\pi/2}(q) \quad (q > 0), \\ \omega_{\pi/2}(q) \leq \omega < \omega_0(q) \quad (q < 0), \end{aligned} \quad (17)$$

where

$$\omega_0(q) \equiv -\frac{1}{q}, \quad (18)$$

$$\omega_{\pi/2}(q) \equiv \frac{a - q}{2M^2 - q(a - q) + 2M\sqrt{M^2 - (a^2 - q^2)}}. \quad (19)$$

One of the bounds for the angular velocity $\omega_0(q)$ corresponds to $r_{\text{eff}}^{+} \rightarrow \infty$ and $\theta_{\text{eff}}^{+} \rightarrow 0$, and the other $\omega_{\pi/2}(q)$ corresponds to $r_{\text{eff}}^{+} = M + \sqrt{M^2 - (a^2 - q^2)}$ and $\theta_{\text{eff}}^{+} = \pi/2$. If $q = 0$, the bound of the angular velocity of the string, $\omega_{\pi/2}(q)$, and the radius of the effective horizon, r_{eff}^{+} , will coincide with the angular velocity Ω_h and radius r_h of the event horizon on the equatorial plane, respectively. When $\omega = 0$, the radius of the effective horizon coincides with that of the ergosphere, namely $r_{\text{eff}}^{+}(0, q) = r_{\text{ergo}}(\theta_{\text{eff}}^{+})$.

It is worth noting that, because we can find that the sign of $\partial r_{\text{eff}}^{+}/\partial \omega$ depends only on the sign of q (see Appendix A 1), both of $r_{\text{eff}}^{+}(\omega, q)$ and $\theta_{\text{eff}}^{+}(\omega, q)$ are monotonic functions of ω for a given q . The regions determined by Eq. (17) are whole regions allowed for $(r_{\text{eff}}^{+}, \theta_{\text{eff}}^{+})$ to exist (we will later depict the regions in Figs. 1 and 3).

B. Equations of motion and boundary conditions

We have seen the position of the string in the spacetime is determined so that the configuration of the string should regularly extend over the effective horizon on the induced geometry. We will show that the regularity conditions can determine the first derivative, also. It follows that we can obtain boundary conditions enough to solve the equations of motion for the string at the effective horizon. Once we have required rigidly rotating strings to enter into the effective horizon, their boundary conditions are determined.

The action (8) still has the degree of freedom arising from coordinate transformation of the spatial worldsheet coordinate σ . This means that one can choose σ for one's own convenience. With taking σ as an affine parameter (i.e., $\mathcal{L} = -1$), the equations of motion for $r(\sigma)$, $\theta(\sigma)$, and $\varphi(\sigma)$ are

$$\frac{F\Sigma}{\Delta} r'' + \frac{1}{2} \left(\frac{F\Sigma}{\Delta} \right)_{,r} r'^2 + \left(\frac{F\Sigma}{\Delta} \right)_{,\theta} r' \theta' - \frac{(F\Sigma)_{,r}}{2} \theta'^2 - \frac{\Delta_{,r}}{2} \sin^2 \theta \varphi'^2 = 0, \quad (20)$$

$$F\Sigma \theta'' - \frac{(F\Sigma)_{,\theta}}{2\Delta} r'^2 + (F\Sigma)_{,r} r' \theta' + \frac{(F\Sigma)_{,\theta}}{2} \theta'^2 - \Delta \sin \theta \cos \theta \varphi'^2 = 0, \quad (21)$$

and

$$\Delta \sin^2 \theta \varphi' = -q, \quad (22)$$

which corresponds to Eq. (10). Note that we have used a comma to denote a partial derivative with respect to a spacetime coordinate, such as $F_{,r} = \partial_r F$.

Since $F(r, \theta)$ in the equations of motion vanishes at the effective horizon, we should require conditions for the solutions to be regular. At $\sigma = \sigma_0$ such that $r(\sigma_0) = r_{\text{eff}}^\pm$ and $\theta(\sigma_0) = \theta_{\text{eff}}^\pm$, the regularity for the equations of motion yields

$$\left(\frac{F_{,r}}{2} r'^2 + F_{,\theta} r' \theta' - \frac{F_{,r}}{2} \Delta \theta'^2 \right) \Big|_{\sigma=\sigma_0} = \frac{\Delta_{,r}}{2\Sigma} \Big|_{\sigma=\sigma_0}, \quad (23)$$

$$\left(-\frac{F_{,\theta}}{2\Delta} r'^2 + F_{,r} r' \theta' + \frac{F_{,\theta}}{2} \theta'^2 \right) \Big|_{\sigma=\sigma_0} = \frac{\sqrt{\Delta - q^2}}{q\Sigma} \Big|_{\sigma=\sigma_0}, \quad (24)$$

where we have used $\sin \theta_{\text{eff}}^\pm = |q|/[\Delta(r_{\text{eff}}^\pm)]^{1/2}$. If $(\Delta_{,r} F_{,r} - 2F_{,\theta} \cot \theta)|_{\sigma=\sigma_0} > 0$, we obtain boundary conditions for the first derivatives of $r(\sigma)$ and $\theta(\sigma)$ as

$$r'^2|_{\sigma=\sigma_0} = \left[\frac{\Delta_{,r} F_{,r} - 2F_{,\theta} \cot \theta}{2\Sigma(F_{,r}^2 + F_{,\theta}^2 \Delta^{-1})} + \frac{1}{2\Sigma} \sqrt{\frac{\Delta_{,r}^2 + 4\Delta \cot^2 \theta}{F_{,r}^2 + F_{,\theta}^2 \Delta^{-1}}} \right] \Big|_{\sigma=\sigma_0}, \quad (25)$$

$$\theta'^2|_{\sigma=\sigma_0} = \left[-\frac{\Delta_{,r} F_{,r} - 2F_{,\theta} \cot \theta}{2\Delta\Sigma(F_{,r}^2 + F_{,\theta}^2 \Delta^{-1})} + \frac{1}{2\Delta\Sigma} \sqrt{\frac{\Delta_{,r}^2 + 4\Delta \cot^2 \theta}{F_{,r}^2 + F_{,\theta}^2 \Delta^{-1}}} \right] \Big|_{\sigma=\sigma_0}, \quad (26)$$

where the relative sign between $r'(\sigma_0)$ and $\theta'(\sigma_0)$ should be positive because of $\Sigma(F_{,r}^2 + \Delta F_{,\theta}^2) r' \theta'|_{\sigma=\sigma_0} = (\Delta_{,r} F_{,\theta} + 2\Delta F_{,r} \cot \theta)|_{\sigma=\sigma_0} > 0$. Note that we can solve the equations of motion outward or inward from the effective horizon $\sigma = \sigma_0$ when we choose $r'(\sigma_0) > 0$ or $r'(\sigma_0) < 0$ respectively.

C. Parameter regions of physical solutions: energy extraction

Provided that parameters ω and q are given in the allowed regions, we have two sets of the boundary conditions $\{r, \theta, r', \theta'\}$ for each branch of r_{eff}^\pm and can obtain regular string solutions at least near the effective horizon. Since the angular momentum flux q and the energy flux ωq have been defined to be positive when their directions are outward, $q > 0$ and $\omega q > 0$ can respectively describe processes of extracting the angular momentum and the energy from the black hole. If one is interested in the energy extraction for example, the string solutions with $\omega q > 0$ are important. However, even though the string configurations are regular, all of them cannot describe physical processes. Only one of the branches of solutions can describe physically reasonable process, while the other describes unphysical but time-reversal process.

As a simple and intuitive manner to discriminate the physical process, we will argue total thermodynamic system constituted of the string and the rotating black hole. We suppose the total energy and total angular momentum conservation are satisfied for the total system. Then, we have

$$\frac{dM_{\text{BH}}}{dt} + \omega q = 0, \quad \frac{dJ_{\text{BH}}}{dt} + q = 0, \quad (27)$$

where M_{BH} and J_{BH} are the mass and angular momentum of the black hole. The first law of black hole thermodynamics⁴ yields

$$\frac{dS_{\text{BH}}}{dt} = \frac{1}{T} \left(\frac{dM_{\text{BH}}}{dt} - \Omega_{\text{h}} \frac{dJ_{\text{BH}}}{dt} \right) = \frac{q}{T} (\Omega_{\text{h}} - \omega), \quad (28)$$

where T is the Hawking temperature. Since physical processes should satisfy $dS_{\text{BH}}/dt \geq 0$, this implies that $\omega \leq \Omega_{\text{h}}$ for $q \geq 0$ and $\omega \geq \Omega_{\text{h}}$ for $q \leq 0$. Therefore, the physically reasonable solutions seem to be those of the r_{eff}^+ -branch.

⁴ The first law of black hole thermodynamics for the Kerr black hole is given by $dM = TdS + \Omega_{\text{h}}dJ$, where the temperature T , entropy S , angular momentum J , and angular velocity Ω_{h} are defined by

$$T = \frac{\Delta_{,r}(r_{\text{h}})}{4\pi(r_{\text{h}}^2 + a^2)}, \quad S = 2\pi M r_{\text{h}}, \quad J = aM, \quad \Omega_{\text{h}} = \frac{a}{r_{\text{h}}^2 + a^2}.$$

The induced geometry on the string worldsheet offers further insights into criteria for determining physical process. Now, (t, σ) -component of the induced metric can be rewritten as

$$h_{t\sigma} = \frac{\varphi' \sin^2 \theta}{\Sigma} \{ \Delta a (1 - \omega a \sin^2 \theta) - (r^2 + a^2) [a - \omega(r^2 + a^2)] \}. \quad (29)$$

At the effective horizon, $\sigma = \sigma_0$, we evaluate it as

$$h_{t\sigma}|_{\sigma=\sigma_0} = -\mathcal{L} \frac{q[a - (a^2 + r^2)\omega]}{\Delta(r)(1 - \omega a \sin^2 \theta)} \Big|_{\sigma=\sigma_0}. \quad (30)$$

Since $\mathcal{L} < 0$, $\Delta(r_{\text{eff}}^\pm) > 0$, and $1 - \omega a \sin^2 \theta_{\text{eff}}^\pm > 0$ (see Appendix A 3 for detail), we have $h_{t\sigma} > 0$ for $q[a - (a^2 + r^2)\omega] > 0$ and $h_{t\sigma} < 0$ for $q[a - (a^2 + r^2)\omega] < 0$ at the effective horizon. This indicates that solutions of the r_{eff}^+ -branch have $h_{t\sigma} > 0$ and those of the r_{eff}^- -branch have $h_{t\sigma} < 0$ at the effective horizon. The sign of $h_{t\sigma}$ is associated with whether the Killing horizon generated by $\xi^a = (\partial_t)^a$ on the worldsheet becomes black-hole type for $h_{t\sigma} > 0$ or white-hole type for $h_{t\sigma} < 0$. We can simply understand it as follows. At the effective horizon, two future-directed null vectors with respect to the induced metric are given by

$$\xi^a = (\partial_t)^a, \quad \chi^a = h_{\sigma\sigma} (\partial_t)^a - 2h_{t\sigma} (\partial_\sigma)^a. \quad (31)$$

It turns out that ξ^a is out-going null vector and χ^a is in-going one if $h_{t\sigma} > 0$, whereas ξ^a is in-going null vector and χ^a is out-going one if $h_{t\sigma} < 0$. Coordinates on the worldsheet can be freely chosen, and dynamics of the string dose not explicitly depend on the worldsheet coordinates, so that a time coordinate on the worldsheet has little physical meaning in general. However, because we have identified the time coordinate on the worldsheet with the Boyer-Lindquist time t on the spacetime, time evolutions on the worldsheet described by this time t have physical meanings. When ξ^a associated with the physical time t generates the white-hole like effective horizon, we should provide information at the effective horizon whenever we solve time evolutions in terms of t . It means that these configurations will be never realized unless specific conditions continue to be provided. On the other hand, when ξ^a generates the black-hole like effective horizon, we do not need any conditions at the effective horizon to solve time evolutions because it is located in the causal future. Therefore, such configurations can be naturally realized by time evolutions without any specific initial conditions or any information beyond the stationary region.

Parameter spaces for the rigidly rotating strings of the r_{eff}^+ -branch are shown in Fig. 1. Allowed regions in (ω, q) -space, which is given by Eq. (17), are enclosed by $\omega_0(q)$ and $\omega_{\pi/2}(q)$. In the r_{eff}^+ -branch, the energy-extraction process, $\omega q > 0$, corresponds to the region $0 < \omega \leq \omega_{\pi/2}(q)$ for $q > 0$. If $q > 0$, $r_{\text{eff}}^+(\omega, q)$ is monotonically decreasing in terms of ω because $\partial r_{\text{eff}}^+ / \partial \omega < 0$ (see Appendix A 1). As we have mentioned, the effective horizon coincides with the ergosphere when $\omega = 0$, that is, $r_{\text{eff}}^+(0, q) = r_{\text{ergo}}(\theta_{\text{eff}}^+)$. It means that r_{eff}^+ is always less than r_{ergo} in the interval $0 < \omega \leq \omega_{\pi/2}(q)$ for $q > 0$. In addition, $\omega_{\pi/2}(q)$ can be rewritten as

$$\omega_{\pi/2}(q) = \frac{2Ma}{r^3 + a^2r + 2Ma^2} - q \frac{r}{r^3 + a^2r + 2Ma^2}, \quad (32)$$

where r should satisfy $\Delta(r) = q^2$. Since the first term monotonically decreases for $r \geq r_h$ and equals Ω_h when $r = r_h$, we have $\omega_{\pi/2}(q) \leq \Omega_h$ for $q \geq 0$ with equality if and only if $q = 0$. As a result, we can conclude that a necessary condition for the energy extraction is that the effective horizon on the string should enter the inside of the ergoregion and the angular velocity of the string should be less than that of the black hole. It is obvious that the parameter region where the energy extraction can occur becomes wider as the Kerr parameter a is larger, and it vanishes for $a = 0$. In the (ω, q) -plane the power of the extraction ωq is described by the area of the rectangle whose sides are ω -axis and q -axis. For a fixed ω , the power will become larger as θ_{eff}^+ closes to the equatorial plane. For a fixed θ_{eff}^+ , it will be maximized as ω becomes about half of the black hole angular velocity Ω_h . Figure 2 shows explicit examples of the string configurations in the parameter region where the energy extraction occurs. We can see that the strings extend sufficiently far from the black hole, while the effective horizons of them are located inside the ergoregion.

The effective horizon on the induced geometry emerges when the rigidly rotating string regularly pass through a light surface on the spacetime. For a given ω , the effective horizon is uniquely determined, whereas there are several possibilities for the light surfaces; the inner light sphere near the black hole and the outer light cylinder exist, or the single connected light surface does. Therefore, even in the r_{eff}^+ -branch one should be careful about which light surface the string is passing through. We find critical angular velocities ω_c^\pm ($\omega_c^- < 0 < \omega_c^+$), at which the inner and outer light surfaces merge on the equatorial plane, is given by the extrema of $\omega_{\pi/2}(q)$, namely $\omega_c^\pm = \omega_{\pi/2}(q_c^\pm)$ such that $d\omega_{\pi/2}/dq|_{q=q_c^\pm} = 0$ (see Appendix A 2 for detail). The topology of the light surface changes at $\omega = \omega_c^\pm$, while the configurations of the string may continuously deform in terms of the two parameters (ω, q) . Hence, the parameter

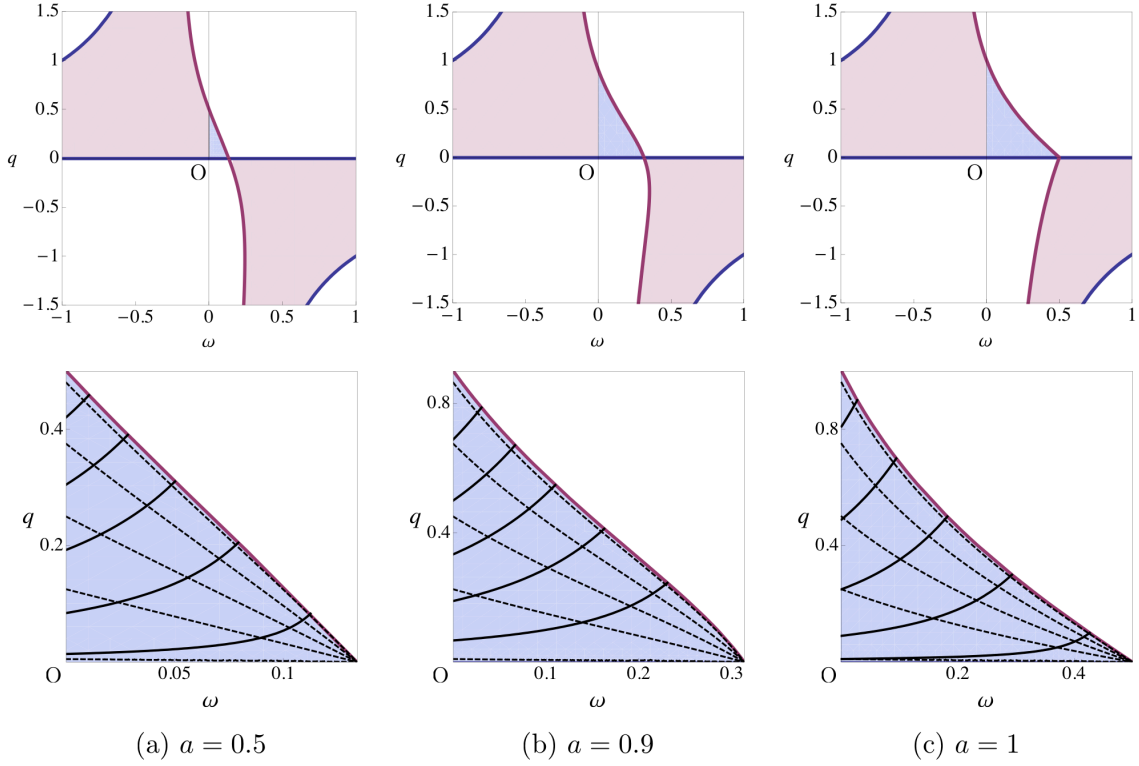


FIG. 1: Parameter spaces for the stationary strings that pass through the effective horizon regularly in the case of (a) $a = 0.5$ (left), (b) $a = 0.9$ (center), and (c) $a = 1.0$ (right) in the unit $M = 1$, which are shown by the shaded regions in the upper plots. Each region is enclosed by $\omega_0(q)$ (blue solid lines) and $\omega_{\pi/2}(q)$ (red solid lines). The lower plots show contours of r_{eff}^+ and θ_{eff}^+ in the case where positive energy extraction occurs. The dashed lines denote the contours of $\sin \theta_{\text{eff}}^+ = 0.98, \sqrt{3}/2, 1/\sqrt{2}, 1/2, 0.1$ from the top to the bottom in each plot. The solid gray lines denote the contours of (a) $r_{\text{eff}}^+ = 2, 1.98, 1.95, 1.92, 1.89, 1.87$; (b) $r_{\text{eff}}^+ = 1.9, 1.8, 1.7, 1.6, 1.5, 1.4$; (c) $r_{\text{eff}}^+ = 1.9, 1.7, 1.5, 1.3, 1.1$, from the top to the bottom.

region can be classified into the following three categories based on the light surface passed by the string: (i) the inner light sphere near the black hole when $\omega_c^- \leq \omega \leq \omega_c^+$ and $q_c^+ \leq q \leq q_c^-$, (ii) the outer light cylinder when $\omega_c^- \leq \omega \leq \omega_c^+$ and $q \leq q_c^+, q \geq q_c^-$, and (iii) the connected light surface when $\omega \leq \omega_c^-, \omega \geq \omega_c^+$. In addition, the points (ω_c^\pm, q_c^\pm) are the critical points at which the three regions join. A sketch of the parameter region is depicted in Fig. 3. It is worth noting that in the region for $\omega q > 0$, which we have been interested in, the effective horizon on the string is the inner light sphere. We can confirm that, when the energy extraction occurs, the string is regularly passing through the inner light sphere and is twining around the black hole.

This result indicates that the energy extraction can occur when the string enter the inside of the ergoregion, and relations between the event horizon and the string configuration such as whether the string would intersect with the event horizon are irrelevant. This is not surprising logically. Outward flux conveying energy to infinity, physically, cannot propagate with exceeding the speed of light. Such energy flux must depend only on its causal past. By the definition of the event horizon, the causal past of energy flux which can reach to infinity does not include the event horizon. Hence, the event horizon is irrelevant to occurrence of the energy extraction by outward energy flux.

III. GLOBAL CONFIGURATIONS

In this section, we discuss global configurations of the regular string extracting the rotational energy from a Kerr black hole in two regimes: One is a slow-rotation regime $a \ll M$ and the other is the extremal case $a = M$.

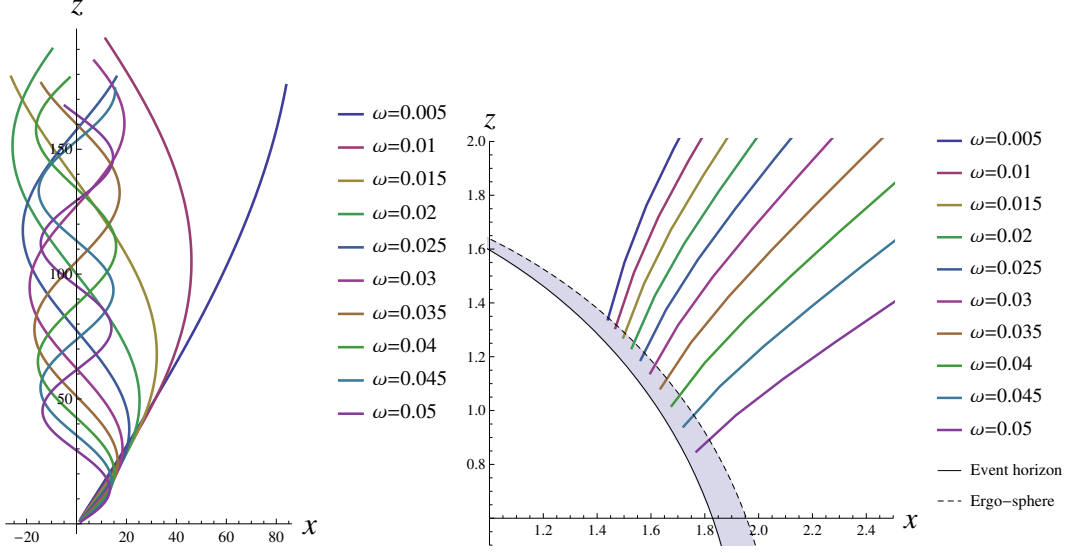


FIG. 2: Examples of the string configurations which can extract positive energy from the black hole for $M = 1$, $a = 1/2$ and $q = 1/4$. The left panel shows global configurations of the strings for various angular velocities ω . As ω is larger, the pitch along z -axis tends to be shorter. The right panel shows an enlarged view near the black hole. Each end-point of the string inside the ergoregion (the shaded region) corresponds to the locus of the effective horizon. As ω is larger, the locus of the effective horizon tends to be close to the equatorial plane. Coordinates have been taken as $(x, z) = (\sqrt{r^2 + a^2} \sin \theta \sin \phi, r \cos \theta)$.

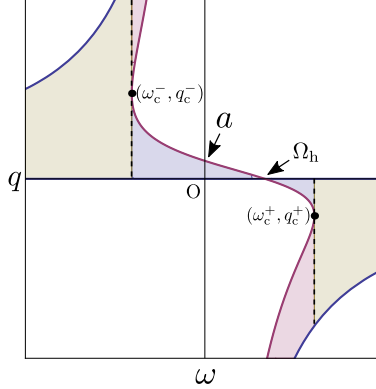


FIG. 3: Sketch of the parameter region for a typical a ($0 < a < 1$). The effective horizon on the string respectively corresponds to (i) the inner light surface when $\omega_c^- \leq \omega \leq \omega_c^+$ and $q_c^- \leq q \leq q_c^+$, (ii) the outer light cylinder when $\omega_c^- \leq \omega \leq \omega_c^+$ and $q \leq q_c^-, q \geq q_c^+$, and (iii) the connected light surface when $\omega \leq \omega_c^-, \omega \geq \omega_c^+$. The critical point (ω_c^\pm, q_c^\pm) , where the three regions join, is the extremum of $\omega_{\pi/2}(q)$.

A. Slow-rotation approximation

For general string configurations with $\omega \neq 0$, we do not have analytic solutions of the equations of motion. To study analytically some global properties of rigidly rotating string configurations, we perform perturbation analysis of the equations of motion by supposing that the rotation of the Kerr black hole is slow, namely $a \ll M$. This perturbation analysis can be regarded as extensions of the work [17] to rigidly rotating string configurations with $\omega \neq 0$ in the slow-rotation regime. We use the same gauge with the one adopted in Ref. [17] for usefulness of comparison with it. Thus the string configuration is given by the embedding $t = \tau + \eta(r)$, $\phi = \omega t + \varphi(r)$, and $\theta = \theta(r)$.⁵ The resultant

⁵ Note that $\eta(r)$ means a gauge choice for a time coordinate τ on the worldsheet, and one can choose $\eta(r)$ so that the induced metric becomes diagonal in these worldsheet coordinates, for example. However, the following equations of motion by using only quantities

equations of motion are

$$\left(\frac{d\varphi}{dr}\right)^2 = \frac{Gq^2}{\Delta^2 \sin^4 \theta}, \quad (33)$$

$$\frac{1}{\sqrt{G}} \frac{d}{dr} \left(\frac{\Sigma F}{\sqrt{G}} \frac{d\theta}{dr} \right) = \frac{\cos \theta}{\Delta \sin^3 \theta} Z, \quad (34)$$

where

$$Z = q^2 - (q^2 - \Delta \sin^2 \theta) \left(1 - \frac{\Delta(1 - a^2 \omega^2 \sin^4 \theta)}{\Sigma F} \right), \quad (35)$$

$$G = \frac{\Sigma F \sin^2 \theta}{\Delta \sin^2 \theta - q^2} \left[1 + \Delta \left(\frac{d\theta}{dr} \right)^2 \right]. \quad (36)$$

Note that Δ , Σ , and F have been defined in the previous section. We can reduce Eqs. (20)–(22) to the above equations of motion with a transformation of variables such as $d\theta/dr = \theta'/r'$, $d\varphi/dr = \varphi'/r'$, and so on. The solutions of Eqs. (33) and (34) for $\omega = 0$ which are regularly crossing the ergosphere were obtained analytically in Ref. [16]. They are explicitly written as

$$\theta(r) = \theta_0, \quad \varphi_{\pm}(r) = \pm \frac{a}{2\sqrt{M^2 - a^2}} \log \left(\frac{r - r_h + 2\sqrt{M^2 - a^2}}{r - r_h} \right), \quad (37)$$

where θ_0 is the integration constant related to q as $q = \pm a \sin^2 \theta_0$ and a constant term within $\varphi_{\pm}(r)$ has been discarded thanks to the axisymmetry.

Here we consider string configurations with $\omega \neq 0$ in the slow-rotation regime. We can define two regions in the Kerr spacetime as

$$\begin{aligned} \text{far region : } r - r_h &\gg M, \\ \text{near region : } r - r_h &\ll M^2/a. \end{aligned} \quad (38)$$

Note that the slow-rotation regime $a \ll M$ allows us to take an overlap region between the far and near regions as $M \ll r - r_h \ll M^2/a$. Furthermore we assume the following scalings for ω and q :

$$\omega = O(a/M^2), \quad q = O(a), \quad (39)$$

which mean that the strings, also, are slowly rotating. In this setting, we can neglect the black hole and regard the spacetime as a flat spacetime in the far region at the leading order. At the near region, we can solve Eqs. (33) and (34) by perturbation analysis in the a/M -expansions. The outer light cylinder is not located in the near region and the inner light sphere is not in the far region, so that we should obtain regular solutions in each region and match the near region solutions with the far region solutions in order to have globally regular solutions.

1. Far region

At the far region the equations of motion reduce to those in the flat spacetime. The general solutions of Eqs. (33) and (34) in the flat spacetime were obtained by Ref. [17] in cylindrical coordinates as

$$z(\rho) = \frac{p}{2\omega} \left(\arcsin \frac{B - 2\omega^2 \rho^2}{C} + z_0 \right), \quad (40)$$

$$\varphi(\rho) = \pm \frac{1}{2} \left(\arcsin \frac{B\rho^2 - 2q^2}{C\rho^2} + \omega q \arcsin \frac{B - 2\omega^2 \rho^2}{C} + \varphi_0 \right), \quad (41)$$

where

$$B \equiv 1 - p^2 + q^2 \omega^2, \quad C \equiv \sqrt{B^2 - 4\omega^2 q^2}, \quad (42)$$

associated with the spacetime coordinates are independent of the gauge choice on the worldsheet.

and z_0 , φ_0 , and p are integration constants. The coordinates z and ρ can be identified with $z = r \cos \theta$ and $\rho = \sqrt{r^2 + a^2} \sin \theta$ in the Boyer-Lindquist coordinates, respectively. The constant p corresponds to the momentum flux per unit length along the cylinder, and the constants z_0 and φ_0 are appropriate offsets for the positions. They should be determined by matching with near-region solutions later. The string in the flat spacetime can have only one effective horizon, which we identify with the outer light cylinder, at $\rho = \omega^{-1}$ since $F = 1 - \omega^2 \rho^2$. The string configuration in the flat spacetime given by the solutions (40) and (41) is constrained to lie in $\rho_- \leq \rho \leq \rho_+$, where ρ_{\pm} defined by

$$\rho_{\pm} \equiv \frac{1}{\omega} \sqrt{\frac{B \pm C}{2}} \quad (43)$$

are turning points of string configurations.⁶ If we require the regularity condition on the outer light cylinder, we have $\rho_+ = \rho_- = \omega^{-1}$ with $q\omega = 1$. Such regularity condition constraints the configuration of regular strings completely on the outer light cylinder, and it cannot be extended from the outer light cylinder. We are now interested in the string configurations which extend into the inside of the outer light cylinder and reach the inner light sphere to extract the rotation energy of the Kerr black hole. Therefore, we will not impose the regularity condition at the outer light cylinder, and instead, consider solutions satisfying

$$\rho_+ < \omega^{-1}. \quad (44)$$

Such solutions never reach the outer light cylinder, and their string configurations can be regular without satisfying the regularity condition on the outer light cylinder. The condition (44) is regarded as that for p . As we will see below, p obtained by matching actually satisfies the condition.

2. Near region

Next, in the near region, we perturbatively solve the equations of motion (33) and (34) in the a/M -expansions. With $a = 0$ and $\omega = 0$ the solution of Eqs. (33) and (34) is given by

$$\theta(r) = \theta_0, \quad \varphi(r) = 0, \quad (45)$$

which is derived from Eq. (37) with $a = 0$ and leads to $q = 0$. Assuming this solution as a zeroth-order solution, we expand $\varphi(r)$, $\theta(r)$, q , and ω as

$$\theta(r) = \theta_0 + \theta_2(r) \left(\frac{a}{M}\right)^2 + \theta_4(r) \left(\frac{a}{M}\right)^4 + \cdots, \quad \varphi(r) = \varphi_1(r) \left(\frac{a}{M}\right) + \varphi_3(r) \left(\frac{a}{M}\right)^3 + \cdots, \quad (46)$$

$$q = q_1 \left(\frac{a}{M}\right) + q_3 \left(\frac{a}{M}\right)^3 + \cdots, \quad \omega = \omega_1 \left(\frac{a}{M}\right) + \omega_3 \left(\frac{a}{M}\right)^3 + \cdots. \quad (47)$$

We solve the equations of motion order by order under these expansions with the regularity condition that the solution is regular at the inner light sphere $r = r_{\text{eff}}^{\pm}$. As we have seen, this regularity condition gives the boundary conditions for string configurations at the inner light sphere. Expanding r_{eff}^{\pm} in terms of a/M yields

$$r_{\text{eff}}^{\pm} = 2M - [M \mp q_1(1 - 4M\omega_1)] \frac{a^2}{2M^2} + O(a^4/M^3). \quad (48)$$

Thus the regularity condition, for example, requires that $\theta_2(r)$, $\theta_4(r)$ and so on should be regular at $r = 2M$ in the a/M -expansions. In contrast, since the positions of the inner light surface and the event horizon are degenerate at the zeroth order in the a/M -expansions, the function $\varphi_1(r)$ can be singular in logarithm at $r = 2M$ in the Boyer-Lindquist coordinates due to the rotation effect of the black hole. Thus we need not impose the regularity of the functions $\varphi_1(r)$,

⁶ This constraint is equivalent to the constraints for the argument of arcsin in Eqs. (40) and (41), namely $-1 \leq (B - 2\omega^2 \rho^2)/C \leq 1$ and $-1 \leq (B\rho^2 - 2q^2)/(C\rho^2) \leq 1$.

$\varphi_3(r)$ and so on, at $r = 2M$. Then, by solving Eqs. (33) and (34) with the regularity condition in the a/M -expansions, at the leading order we have regular solutions in the near region as

$$\varphi_1^\pm(r) = \varphi_1^0 - \frac{q_1^\pm [\log(r - 2M) - \log r]}{2M \sin^2 \theta_0}, \quad (49)$$

$$\begin{aligned} \theta_2(r) = \theta_2^0 - \frac{5Mr\omega_1^2 \sin 2\theta_0}{6} - \frac{r^2\omega_1^2 \sin 2\theta_0}{12} - \frac{11M^2\omega_1^2 \sin 2\theta_0}{3} \log \frac{r}{2M} \\ - M\omega_1(1 - 2M\omega_1) \sin 2\theta_0 \left[\text{Li}_2(1 - r/2M) + \frac{1}{2} \left(\log \frac{r}{2M} \right)^2 \right], \end{aligned} \quad (50)$$

where φ_1^0 and θ_2^0 are integration constants and $\text{Li}_2(x)$ denotes the polylogarithm function. For simplicity, we will omit the constant terms φ_1^0 and θ_2^0 hereafter, because they can be absorbed by the constant terms in the zeroth-order solutions. The second and third terms in the right-hand side of Eq. (50) imply the break down of the near-region solution at $r = O(M^2/a)$. The regularity condition for $\theta_2(r)$ at the inner light sphere requires

$$q_1 = q_1^\pm \equiv \pm M(1 - 4\omega_1 M) \sin^2 \theta_0. \quad (51)$$

In Eq. (50) we have already used this condition. We note that the above condition is identical to the regularity condition $q^2 = \Delta \sin^2 \theta|_{r=r_{\text{eff}}^\pm}$ of Eq. (14) at the effective horizon in the a/M -expansions.

Let us confirm some properties shown in the previous section for the near-region solution under the slow-rotation approximation. The regular near-region solutions with $q_1 = q_1^\pm$ correspond to the r_{eff}^\pm -branches. The condition (51) yields

$$q = \pm 4M^2 (\Omega_h - \omega) \sin^2 \theta_0 + O(a^3/M^2), \quad (52)$$

where we have used $\Omega_h = a/(4M^2) + O(a^3/M^4)$. For $\omega < \Omega_h$, the r_{eff}^+ -branch has a positive q and the r_{eff}^- -branch has a negative q . As we have seen, the physically reasonable solution is given by $q_1 = q_1^+$, and $q_1 = q_1^-$ gives its time-reversal solution. Indeed we can see that the physically reasonable solution has the positive energy flux at the inner light surface only when the inner light surface is located in the ergoregion. The energy flux at the inner light surface is given by

$$\begin{aligned} \omega q &= \frac{\omega_1 a^2}{M} (1 - 4M\omega_1) \sin^2 \theta_0 + O(a^4/M^4) \\ &= 4M^2 \omega (\Omega_h - \omega) \sin^2 \theta_0 + O(a^4/M^4). \end{aligned} \quad (53)$$

The locus of the ergosphere is

$$r_{\text{ergo}}(\theta_0) = 2M - \frac{a^2 \cos^2 \theta_0}{2M} + O(a^3/M^2). \quad (54)$$

Using Eq. (51) we have

$$\begin{aligned} r_{\text{ergo}}(\theta_0) - r_{\text{eff}}^+ &= 4a^2 \omega_1 (1 - 2M\omega_1) \sin^2 \theta_0 + O(a^3/M^2) \\ &= 8M^3 \omega (2\Omega_h - \omega) \sin^2 \theta_0 + O(a^3/M^2). \end{aligned} \quad (55)$$

Thus the positive energy flux is realized only if $r_{\text{ergo}} > r_{\text{eff}}^+$. Note that the solutions in the r_{eff}^+ -branch with $2 > 4M\omega_1 > 1$ have the negative energy flux although their inner light surfaces are inside the ergoregion. This is because such solutions are rotating faster than the Kerr black hole and supplying the angular momentum to the black hole. One interesting observation on the energy flux is the fact that the maximum energy flux is realized when the angular velocity of the string is half of the horizon angular velocity as $\omega = \Omega_h/2$ for a given θ_0 . This is the same situation with the Blandford-Znajek process for force-free magnetosphere [8].

3. Matching

Let us perform the matching of the far-region and near-region solutions to see the global configuration of the solution. The behavior of the far-region solution in the overlap region, $M \ll r - r_h \ll M^2/a$, can be obtained by

using the a/M -expansions to the far-region solution (40) and (41). Then we find that the conditions for the matching are

$$p = \cos \theta_0 + O(a/M), \quad z_0 = \frac{\pi}{2} + O(a/M), \quad \varphi_0 = -\frac{\pi}{2} + O(a/M). \quad (56)$$

Under these conditions we can match the regular near-region solution with the far-region solution consistently. Furthermore, for this value of p , the outer turning point ρ_+ satisfies the condition (44)

$$\rho_+ = \frac{M \sin \theta_0}{a\omega_1} < \frac{M}{a\omega_1} \quad (57)$$

except for the string on the equatorial plane $\theta_0 = \pi/2$. This means that the solution with $0 < \theta_0 < \pi/2$ can start from the inner light sphere which is located inside the ergoregion, and extend to the infinity without reaching the outer light cylinder in the slow-rotation limit. As a result, we can extract the rotational energy of the Kerr black hole to the infinity by the rigidly rotating regular strings. This is the result just obtained by perturbation analysis in the slow-rotation approximation. However, we will see that similar properties hold also even in not slow-rotation regime below, and it means that the extraction of the rotational energy from the Kerr black hole without touching the outer light cylinder by the rigidly rotating regular strings can generally occur.

B. The extremal Kerr background

We focus on the global configuration of the stationary rotating string extracting the rotational energy from the extremal Kerr black hole. As shown in Sec. II, since the area in (ω, q) space in which the string extracts positive energy becomes maximum in $a = M$, then we expect to see the difference of the string configuration obviously due to the choice of the parameters.

Figure 4 shows the global configurations of the stationary rotating string on a constant- t slice that carries positive energy from the extremal black hole. Inside the corresponding parameter region, we have selected the six sets of the values $(\omega M, q/M) = (0.2, 0.4), (0.35, 0.15), (0.08, 0.6), (0.08, 0.4), (0.2, 0.15),$ and $(0.08, 0.15)$, which are in order of decreasing the amount of the energy flux. Note that the string with $(\omega M, q/M) = (0.2, 0.4)$ is the closest to the string with the maximum efficiency of the energy extraction. The string in each figure crosses over the effective horizon at $(r, \theta, \phi) = (r_{\text{eff}}^+, \theta_{\text{eff}}^+, 0)$, which is shown as the connection of two colored lines, and extends from the event horizon to a far region. We can conclude that for each set of the parameters the string is twining around the black hole⁷ and extending to the infinity along the rotational axis when the energy extraction occurs.

Let us introduce the cylindrical coordinates (ρ, ϕ, z) that is defined as $\rho = \sqrt{r^2 + a^2} \sin \theta$. The global configuration of the string apart from the black hole is characterized by three characteristics: the “size” in ρ -direction; the “pitch” in z -direction; and the “frequency” of $\rho(\sigma)$ in terms of ϕ , which we have defined as: the proper length in ρ -direction at the maximum value of $\rho(\sigma)$; the proper length in z -direction during one period of the oscillation of $\rho(\sigma)$; and the number of the oscillation of $\rho(\sigma)$ during one period of the oscillation of $\phi(\sigma)$, respectively.

These characteristics are related to the parameters (ω, q) as follows. As seen in Fig. 4, the “size” becomes smaller with increasing ω or decreasing q . The “pitch” in z -direction becomes shorter with increasing ω . The “frequency” of $\rho(\sigma)$ in terms of ϕ decreases as ω or q increases.

When the string goes apart from the black hole, dynamics of the string seems to be well described by that in the flat spacetime as well as in the slow-rotation case. In fact, the behaviors of the string in Fig. 4 are similar to those in the flat spacetime in Ref. [18]. Let us examine the above characteristics by using the general solutions of the rigidly rotating string in the flat spacetime, which have been already shown in Eqs. (40) and (41). Since the co-rotating Killing vector exists among the Kerr and flat spacetime, ω and q in the parameters characterizing the solutions can be identified in the both spacetimes. However, the translational vector along the z -direction is no longer a Killing vector in the Kerr spacetime, so that a quantity corresponding to p cannot be conserved. As seen previously, p in a far region should be determined by solving the equations of motion near the black hole. Because the rigidly rotating

⁷ On constant- t slices in the Boyer-Lindquist coordinates the string seems to be twining around the event horizon endlessly without crossing it, that is, $\varphi(\sigma)$ will diverge as $r(\sigma)$ goes to r_h . However, it does not mean that the string can never penetrate the event horizon because these time-slices intersect with the event horizon only at the bifurcation surface. In fact, the logarithmic divergences which have appeared in the exact solution (37) and the approximate solution (49) for $\varphi(\sigma)$ originate from the regular behavior of $\Delta d\varphi/dr$ at $r = r_h$, so that the string can be across the event horizon on other time-slices in horizon-penetrating coordinates such as the Eddington-Finkelstein coordinates. In general, we expect that the physically reasonable solutions in the r_{eff}^+ -branch can naturally penetrate the (future) event horizon while they cannot be across the (past) white hole horizon.

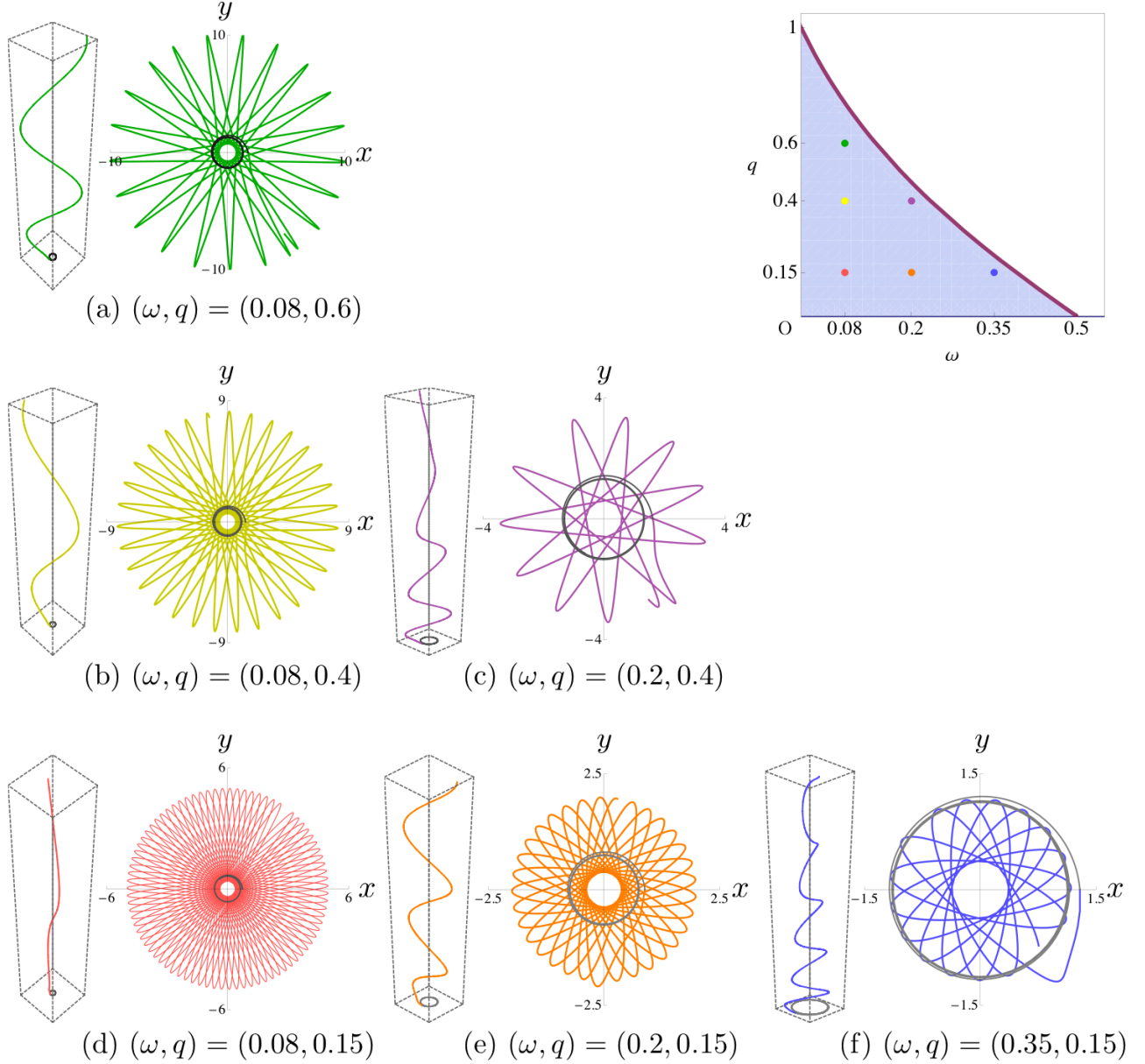


FIG. 4: Snapshots of a stationary rotating string twining around the extremal Kerr black hole, $a = 1$, in units where $M = 1$. The left in each figure is a three-dimensional snapshot of the string at constant- t slice, where the plot range is $0 \leq z \leq 70$ from the bottom to the top. The right in each figure is the string configuration projected onto a constant- z slice. The solid lines colored with gray show the string configuration in the inside of the effective horizon, which are twining around the event horizon. The solid lines colored with green, yellow, purple, red, orange, and blue show the string configuration in the outside of the effective horizon. The plot in the upper right corner gives the parameter region of (ω, q) with which positive energy extraction by the string occurs, where each colored point identified with that of each string indicates to the parameter value in Figs. (a)–(f). Moreover, the arrangement of the points in the parameter region is identified with that of the Figs. (a)–(f).

strings have the energy flux $\omega q \lesssim 0.15$ at most when the energy extraction occurs, we will read some characteristic quantities from the exact solutions in the flat spacetime assuming ωq is small. The “size” given by ρ_+ of Eq. (43) becomes $\rho_+ \simeq \sqrt{1 - p^2}/\omega$ and the “pitch” given by the factor in the front of the first term of Eq. (40) becomes p/ω . Since the factor of the second term of Eq. (41) represents a phase-shift, the “frequency” becomes $\sim (\omega q)^{-1}$. If we suppose $p \simeq \cos \theta_{\text{eff}}^+$ following the result in the slow-rotation approximation, the behaviors of the characteristics observed in Fig. 4 seem to be explained.

IV. SUMMARY AND DISCUSSION

We have studied rigidly rotating Nambu-Goto strings in the Kerr spacetime and have shown that the rotational energy of the black hole can be extracted by the strings. The string configurations are characterized by two parameters: the angular velocity ω and the angular momentum flux q . We have considered the string regularly passing through a light surface and have analytically exhibited the allowed region where such strings exist in the parameter space (ω, q) . We have found a necessary condition for the energy extraction is that the effective horizon on the worldsheet caused by the rigid rotation will enter into the ergoregion of the Kerr black hole and the angular velocity of the rigid rotation is less than that of the black hole, namely $\omega < \Omega_h$. Moreover, global configurations of such strings have been examined in a slow-rotating case ($a \ll M$) and the extremal case ($a = M$). In the both cases we have shown the rigidly rotating strings with positive energy flux can start from the inner light sphere inside the ergoregion and extend to the infinity. It turns out that the energy extraction from the black hole can generally occur.

The current mechanism of the energy extraction can be briefly interpreted on the basis of the usual Penrose process in the ergoregion (the analogy of the Penrose process was mentioned in the literature [19]). As we have mentioned, the effective horizon on the worldsheet corresponds to a stationary limit surface with respect to the co-rotating Killing vector with angular velocity ω . Beyond the effective horizon on the worldsheet of the rigidly rotating string, the Killing vector tangential to the worldsheet becomes spacelike. This does not mean that the proper motion of line elements of the string may become superluminal. The line elements cannot follow the superluminal Killing orbit and the interval of the line elements continues to be larger, that is, the string is not stationary but stretching in this region. In general, if a string with a tension is stretching, its potential energy will increase and the string should consume an energy to stretch. However, the situation changes in the ergoregion. Because the Killing energy can be negative in the ergoregion, the string stretching can decrease its energy similar to the fragmentation in the Penrose process for particles. Hence, if the effective horizon on the string enters into the ergoregion, the string can gain an energy by stretching in the ergoregion and extract the energy to the infinity. This extraction mechanism is quite simple and general. For various other string-like objects as well as Nambu-Goto strings, it is expected that these mechanisms of the energy extraction do work well.

It should be emphasized that the necessary condition of the energy extraction is determined locally near the effective horizon on a light surface but irrelevant of global nature such as configurations of the string at the event horizon or the infinity. This result is reasonable in the following respects. This energy-extraction mechanism can be regarded as a complex of different processes spatially and temporally separated; “generation” of an energy in the ergoregion, “transport” of the energy to the infinity, and “disposal” of residues resulting from the energy generation. Precisely speaking, our necessary condition is a condition for “generation” of the energy. In contrast, existence of event horizons is helpful for the “disposal” process because event horizons of black holes are certainly ideal disposal sites, but not so significant for the “generation” process. Furthermore, in realistic situations of the energy extraction from black holes only a single object or phenomenon does not need to play a major role in all of the above processes. For example, even though a rigidly rotating string cannot entirely reach the infinity, the energy extraction will be successful as long as the string can reach sufficiently far from the black hole and then transfer its energy to other objects without falling back to the black hole. As a result, what is most essential and primal in the energy-extraction mechanism is the “generation” process in the ergoregion, and we expect this fact may be true for various mechanisms of extracting the rotational energy of black holes other than strings discussed in this paper.

Finally, let us quantitatively evaluate the energy-extraction rate $dE/dt = \mu q \omega$ for the string with a tension μ . It is worth noting that this energy-extraction rate is irrelevant to the mass scale of the central black hole, which is canceled out, even though the mass scale determines the amount of the rotational energy to extract. Thus, the string tension dominates the energy-extraction rate via this mechanism. We shall restore the fundamental constants G_N and c , and then we have the energy-extraction rate as

$$\frac{dE}{dt} = 5.8 \times 10^{51} \text{erg/s} \left(\frac{G_N \mu / c^2}{1.3 \times 10^{-7}} \right) \left(\frac{q / a c^2}{1/2} \right) \left(\frac{\omega / \Omega_h}{1/2} \right) u(\alpha), \quad (58)$$

where $u(\alpha) \equiv 1 - \sqrt{1 - \alpha^2}$ ($0 \leq u(\alpha) \leq 1$) and $\alpha \equiv a c^2 / G_N M$ is the dimensionless Kerr parameter.⁸ The string tension μc^2 has been normalized by an observational upper bound on the cosmic string tension [20]. Furthermore, it is interesting and suggestive to apply magnetospheres around a rotating black hole. We consider there are magnetic fields B surrounding a rotating black hole. Total magnetic tension $\mu_B c^2$ is roughly estimated by $\mu_B c^2 = 4\pi r_h^2 B^2 / \mu_0$, where μ_0 is the vacuum permeability. Suppose the magnetic field $B \simeq 10^{15} \text{G}$ and the black hole mass $M \simeq 10 M_\odot$, we

⁸ We have assumed the dimension of μ is ML^{-1} and the dimension of a is L , where M and L respectively denote mass and length.

have $G_N \mu_B / c^2 \simeq 10^{-7}$. This tension gives a maximum energy-extraction rate $\sim 10^{51} \text{ erg/s}$ as we have estimated, and this value is comparable to the usual power caused by the Blandford-Znajek process for the same magnetic field and black hole mass. Thus we expect that the magnetic tensions rather than the magnetic pressure may play an essential role in the energy extraction by the Blandford-Znajek process in black hole magnetosphere.

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Appendix A: Derivations of various quantities

1. Locus of the effective horizon

As we have seen in Sec. II, for given ω and q , the conditions for the effective Killing horizon are given by

$$\Delta(r) = \frac{[a - (a^2 + r^2)\omega]^2}{(1 - a\omega \sin^2 \theta)^2} \sin^2 \theta, \quad (\text{A1})$$

and the regularity condition

$$\Delta(r) \sin^2 \theta = q^2, \quad (\text{A2})$$

in Eqs. (7) and (14). If $\Delta(r_{\text{eff}}) = 0$, that is, $r_{\text{eff}} = r_h$, we have $q = 0$ immediately. It results in $\theta_{\text{eff}} = 0$ and arbitrary ω , or $\omega = \Omega_h$ and arbitrary θ_{eff} . Hereafter we assume that $\Delta > 0$, namely the effective horizon is located outside the event horizon $r_{\text{eff}} > r_h$. Eliminating $\sin^2 \theta$ from the above equations, we have

$$q^2(\Delta/q^2 - a\omega)^2 = [a - (a^2 + r^2)\omega]^2. \quad (\text{A3})$$

This is a quartic equation in terms of r and its positive roots will give the radius of the effective horizon. If $q[a - (a^2 + r^2)\omega] > 0$, then two roots of Eq. (A3) are

$$r = \frac{M}{1 + q\omega} \pm \sqrt{\left(\frac{M}{1 + q\omega}\right)^2 - a(a - q)}, \quad (\text{A4})$$

If $q[a - (a^2 + r^2)\omega] < 0$, then two roots of Eq. (A3) are

$$r = \frac{M}{1 - q\omega} \pm \sqrt{\left(\frac{M}{1 - q\omega}\right)^2 - a(a + q)}. \quad (\text{A5})$$

Requiring the conditions (A2) and $0 \leq \sin^2 \theta \leq 1$ we can exclude the roots with the lower sign among the above four roots, as we will see later, so that we have the radius of the effective horizon

$$r_{\text{eff}}^{\pm} \equiv \frac{M}{1 \pm q\omega} + \sqrt{\left(\frac{M}{1 \pm q\omega}\right)^2 - a(a \mp q)}. \quad (\text{A6})$$

For each branch $r = r_{\text{eff}}^{\pm}$, the polar angle of the effective horizon $\theta_{\text{eff}}^{\pm}$ is given by

$$\sin^2 \theta_{\text{eff}}^{\pm} \equiv \frac{q^2}{\Delta(r_{\text{eff}}^{\pm})}. \quad (\text{A7})$$

As an important property of $r_{\text{eff}}^{\pm}(\omega, q)$, we have

$$\frac{\partial r_{\text{eff}}^{\pm}}{\partial \omega}(\omega, q) = \mp q \frac{M r_{\text{eff}}^{\pm}}{(1 \pm q\omega)^2} \left[\left(\frac{M}{1 \pm q\omega} \right)^2 - a(a \mp q) \right]^{-1/2}, \quad (\text{A8})$$

in which q only can change its sign. Thus, it turns out that r_{eff}^{\pm} is a monotonic function of ω for a fixed q .

We show the exclusion of the roots $r = r_{\text{ex}}$, where

$$r_{\text{ex}} \equiv \frac{M}{1 \pm q\omega} - \sqrt{\left(\frac{M}{1 \pm q\omega}\right)^2 - a(a \mp q)}. \quad (\text{A9})$$

When $q = 0$, $r_{\text{ex}} \geq r_{\text{h}}$ holds only if $M = a$. In this case the equation (A3) becomes degenerate and then we have $r_{\text{ex}} = r_{\text{eff}}^{\pm} = r_{\text{h}} = M$. Since we are interested in the roots $r = r_{\text{ex}}$ such that $r_{\text{ex}} \neq r_{\text{eff}}^{\pm}$, we assume $q \neq 0$ hereafter. To satisfy $r_{\text{ex}} > 0$ leads to $a \mp q > 0$ and $M/(1 \pm q\omega) > 0$. Therefore, we have the following inequality

$$a \mp \frac{q}{2} \geq \sqrt{a(a \mp q)} \geq r_{\text{ex}}, \quad (\text{A10})$$

where we have used $(A + B)/2 \geq \sqrt{AB}$ for $A = a$ and $B = a \mp q$ in the former inequality, and $A + B \geq \sqrt{A^2 + B^2}$ for $A = \sqrt{a(a \mp q)}$ and $B = M/(1 \pm q\omega) - r_{\text{ex}}$ in the latter inequality. In addition, $M \geq a$ yields

$$M + \sqrt{M^2 - a^2 + q^2} \geq a + |q| > a \mp \frac{q}{2}. \quad (\text{A11})$$

As a result, we have $M + \sqrt{M^2 - a^2 + q^2} > r_{\text{ex}}$. Since $\Delta(r)$ is a monotonically increasing function of r for $r > r_{\text{h}}$, we have $q^2 > \Delta(r_{\text{ex}})$. This cannot satisfy the regularity condition (A2) clearly, so that we can rule out $r = r_{\text{ex}}$.

2. Critical point

The rigid rotations in a black hole spacetime generally yield two light surfaces, which are the inner light sphere and the outer light cylinder. However, if the angular velocity becomes sufficiently large, the two light surfaces will merge into a single connected light surface. We consider the critical case in which the two light surfaces touch each other.

Now, we focus on the equatorial plane, $\theta = \pi/2$. For a given ω , the radii of the light surfaces are determined by $f(r, \omega) = 0$, where we have defined

$$\begin{aligned} f(r, \omega) &\equiv rF(r, \pi/2) \\ &= -\omega^2 r^3 + (1 - a^2 \omega^2)r - 2M(1 - a\omega)^2. \end{aligned} \quad (\text{A12})$$

The critical radius r_{c} and angular velocity ω_{c} such that the two light surfaces merge are given by the following conditions

$$f(r_{\text{c}}, \omega_{\text{c}}) = 0, \quad \frac{\partial f}{\partial r}(r_{\text{c}}, \omega_{\text{c}}) = 0, \quad (\text{A13})$$

which means that two positive roots of $f(r, \omega) = 0$ should be degenerate.

On the equatorial plane, the radius of the effective horizon, r_{eff} , should satisfy

$$f(r_{\text{eff}}, \omega) = 0, \quad \Delta(r_{\text{eff}}) = q^2. \quad (\text{A14})$$

Since $f(r, \omega)$ is a quadratic function in terms of ω , the equation $f(r, \omega) = 0$ has two roots

$$\omega^{\pm}(r) \equiv \frac{2Ma}{r^3 + a^2 r + 2Ma^2} \pm \frac{r\Delta^{1/2}}{r^3 + a^2 r + 2Ma^2}. \quad (\text{A15})$$

Note that we find $\omega^+(r) > 0$ and $\omega^-(r) \leq \Omega_{\text{h}}$ for $r \geq r_{\text{h}}$. Then, the latter condition $\Delta(r_{\text{eff}}) = q^2$ yields $r = r_{\text{eff}}(q) \equiv M + \sqrt{M^2 - (a^2 - q^2)}$, so that we obtain the relation between ω and q on the equatorial plane ($\theta_{\text{eff}} = \pi/2$)

$$\omega_{\pi/2}(q) = \frac{a - q}{2M^2 - q(a - q) + 2M\sqrt{M^2 - (a^2 - q^2)}}, \quad (\text{A16})$$

as shown in Eq. (19). Here, we have taken $\omega_{\pi/2}(q) = \omega^-(r_{\text{eff}})$ for $q \geq 0$ and $\omega_{\pi/2}(q) = \omega^+(r_{\text{eff}})$ for $q < 0$ in order to belong to the r_{eff}^{\pm} -branch. Differentiating Eq. (A14) with respect to q and evaluating them at the effective horizon, we have the following identities

$$\partial_r f \frac{dr_{\text{eff}}}{dq} + \partial_\omega f \frac{d\omega_{\pi/2}}{dq} = 0, \quad \Delta_{,r} \frac{dr_{\text{eff}}}{dq} = 2q. \quad (\text{A17})$$

As a result, we have

$$\frac{d\omega_{\pi/2}(q)}{dq} = - \left. \frac{2q\partial_r f}{\Delta, r\partial_\omega f} \right|_{(r,\omega)=(r_{\text{eff}},\omega_{\pi/2})}. \quad (\text{A18})$$

Because $\partial_r f = 0$ at the critical point $r = r_c$ as we have seen, we have proven that if the effective horizon coincides with the critical radius, namely $r_{\text{eff}} = r_c$, then $d\omega_{\pi/2}(q)/dq = 0$.

3. Monotonicity and positivity

Now, we shall prove $1 - \omega a \sin^2 \theta > 0$ at the effective horizon $r = r_{\text{eff}}^\pm$. We have

$$\frac{\partial}{\partial \omega} [\Delta(r_{\text{eff}}^\pm) - a\omega q^2] = \mp q \left\{ \Delta(r_{\text{eff}}^\pm) + \frac{2Mr_{\text{eff}}^\pm}{(1 \pm q\omega)^2} \left(r_{\text{eff}}^\pm - M \frac{1 \pm q\omega}{1 \pm q\omega + q^2\omega^2} \right) \left(r_{\text{eff}}^\pm - \frac{M}{1 \pm q\omega} \right)^{-1} \left[\left(\frac{1}{2} \pm q\omega \right)^2 + \frac{3}{4} \right] \right\}, \quad (\text{A19})$$

where each term in the braces is positive.⁹ Thus, $\Delta(1 - \omega a \sin^2 \theta)$ at $r = r_{\text{eff}}^\pm$ is a monotonic function of ω for a fixed q , and whether it is monotonically increasing or decreasing depends on the sign of q . Since $M \geq a$, we have $a^2 - q^2 - 2M^2 < 0$. It leads to

$$\begin{aligned} a^2 - q^2 - 2M^2 < 0 < 2M\sqrt{M^2 - a^2 + q^2} \\ \iff a(a - q) < 2M^2 - q(a - q) + 2M\sqrt{M^2 - a^2 + q^2}. \end{aligned} \quad (\text{A20})$$

Because the right-hand side of the last inequality is positive if $q < 0$, we obtain

$$\omega_{\pi/2}(q) = \frac{a - q}{2M^2 - q(a - q) + 2M\sqrt{M^2 - (a^2 - q^2)}} < \frac{1}{a} \quad \text{for } q < 0. \quad (\text{A21})$$

Furthermore, as we have shown in the previous subsection, we obtain

$$\omega_{\pi/2}(q) = \omega^-(r_{\text{eff}}) \leq \Omega_h \leq \frac{1}{2a} < \frac{1}{a} \quad \text{for } q \geq 0. \quad (\text{A22})$$

As a result, we conclude that

$$[\Delta(r_{\text{eff}}^\pm) - a\omega q^2]|_{\omega=\omega_{\pi/2}(q)} = q^2[1 - a\omega_{\pi/2}(q)] > 0. \quad (\text{A23})$$

Note that, even though $\omega_{\pi/2}(q)$ has been defined only for the r_{eff}^+ -branch, the above result can be applied to the r_{eff}^- -branch because the r_{eff}^\pm -branch is related to each other $q \rightarrow -q$.

From Eqs. (A19) and (A23), we have proven that $1 - \omega a \sin^2 \theta > 0$ at $r = r_{\text{eff}}^\pm$ for arbitrary q .

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⁹ $r_{\text{eff}}^\pm \geq M \geq M(1 \pm q\omega) / (1 \pm q\omega + q^2\omega^2)$

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